Optimal sensor placement for the determination of toroidal harmonic coefficients from magnetic flux density data

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***Abstract*—The toroidal harmonic expansion is suitable to describe magnetic fields in the vacuum domain of strongly curved magnet systems. Its degrees of freedom are called toroidal harmonic coefficients. They can be determined from magnetic flux density data by linear least squares fitting. The conditioning of the linear problem depends on the sensor positions. The objective of this paper is to optimize the sensor positions based on the matrix conditioning. This optimization problem is solved using swarm optimization. The results indicate that the sensors should be placed close to the torus boundary with an almost equidistant distribution regarding the poloidal coordinate of the toroidal coordinate system.**

I. INTRODUCTION

The classical way of expressing the magnetic field in straight accelerator magnets is through the Fourier coefficients of the eigenfunctions of the Laplace equation in polar coordinates. When the magnets are curved, the scaling laws derived for straight magnets are no longer applicable. Instead, the eigenfunctions of the Laplace equation in the toroidal coordinate system must be used, leading to the toroidal harmonic expansion [1]. Different approaches for determining the coefficients of the toroidal harmonic expansion by fitting the expansion to magnetic flux density data can be found in the literature [1], [2]. However, it was not investigated at which positions the magnetic flux density data should be taken to optimize the conditioning of the problem. This question arises naturally when designing a sensor system. In this work, an optimization problem is solved to determine the optimal sensor positions concerning the condition number of the fitting problem. Placing the sensors close to the torus boundary with an equidistant distribution regarding the poloidal angle σ improves the condition number by one order of magnitude compared to an equidistant sensor distribution in space.

II. METHODS

A. Toroidal harmonic expansion

The magnetostatic problem in the vacuum domain can be solved using the scalar potential formulation leading to the Laplace equation $\Delta\phi_{\mathrm{m}} = 0$.

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The toroidal coordinates (τ, σ, φ) are defined by

$$\left(\frac{a\sinh(\tau)\cos(\varphi)}{\cosh(\tau)-\cos(\sigma)}, \quad \frac{a\sinh(\tau)\sin(\varphi)}{\cosh(\tau)-\cos(\sigma)}, \quad \frac{a\sin(\sigma)}{\cosh(\tau)-\cos(\sigma)}\right) \in \mathbb{R}^3 \quad (1)$$

and are suitable for describing points in the solid ring torus $\mathbb{S}^1 \times D^2$ (the product of the 1D sphere $\mathbb{S}^1$ and the unit disk D^2). The coordinate surfaces of τ and σ are the set of Apollonian circles at the focal ring located at radius $a = R\sqrt{\epsilon^2 - 1}$. The aspect-ratio $\epsilon := \rho/R$ is defined by the toroidal radius ρ and the poloidal radius R of the solid torus. The torus boundary is defined by the $\tau = \tau_{\mathrm{out}} := \operatorname{acosh}(\rho/R)$-iso surface.

The Laplace equation can be solved in toroidal coordinates using R-separation [3]. In this paper, we restrict ourselves to studying magnetic flux densities independent of the rotational parameter φ. This case is relevant for homogeneous field regions and φ-periodic flux densities integrated over φ. Neglecting non-physical solutions of the Laplace equation and including a term linear in φ due to the double connectedness of the toroidal domain yields the following expansion of the magnetic scalar potential [1]

$$\begin{aligned}\phi_{\mathrm{m}}(\tau, \sigma, \varphi) \quad = \quad & \mathcal{E}\varphi + \sqrt{k(\tau, \sigma)} \sum_{n=0}^{\infty} Q^0_{n-1/2}(\cosh(\tau)) \\ & (\mathcal{A}_n \cos(n\sigma) + \mathcal{B}_n \sin(n\sigma)) \,. \qquad (2)\end{aligned}$$

In this equation, $\mathcal{E}, \mathcal{A}_n, \mathcal{B}_n \in \mathbb{R}$ are scalar coefficients, we call $\mathcal{A}_n, \mathcal{B}_n$ the *toroidal harmonic coefficients* and equation (2) is referred to as *toroidal harmonic expansion* of the magnetic scalar potential. The factor k originating from the R-separation is given by $k(\tau, \sigma) := \cosh(\tau) - \cos(\sigma)$. The terms $Q^0_{n-1/2}$ are the associated Legendre functions of the second kind of half-integer degree [4], [5]. For numerical purposes, we use the normalization of the toroidal harmonic coefficients introduced by [1] and replace $Q^0_{n-1/2}(\cosh(\tau))$ with $Q^0_{n-1/2}(\cosh(\tau))/Q^0_{n-1/2}(\cosh(\tau_0))$, where $\tau = \tau_0$ is an arbitrary chosen fixed iso-surface.

Computing the gradient of the magnetic scalar potential in the toroidal harmonic expansion yields the magnetic flux density $\mathbf{B}$ in the toroidal harmonic expansion. In contrast to the magnetic scalar potential, the magnetic flux density can be determined from magnetic measurements, e.g., using Hall probes [6]. To save on notation, the full equations of the magnetic flux density are not given in this abstract. However, they can be found, e.g., in [7]. Note that the toroidal harmonic coefficients appear as linear coefficients in the components of the magnetic flux density. Therefore, scalar functions $f^{\mathcal{A}}_{\tau,n}, f^{\mathcal{B}}_{\tau,n}, f^{\mathcal{A}}_{\sigma,n}, f^{\mathcal{B}}_{\sigma,n}, f^{\mathcal{E}}_{\varphi,n}$ dependent on the coordinates (τ,σ), the normalization parameter τ_0 and the focal radius a can be defined such that the components of the magnetic flux density can be written as

$$\mathbf{B}_\tau(\tau,\sigma) = \sum_{n=0}^{\infty}\left[\mathcal{A}_n\, f^{\mathcal{A}}_{\tau,n}(\tau,\sigma) + \mathcal{B}_n\, f^{\mathcal{B}}_{\tau,n}(\tau,\sigma)\right] \quad (3)$$

$$\mathbf{B}_\sigma(\tau,\sigma) = \sum_{n=0}^{\infty}\left[\mathcal{A}_n\, f^{\mathcal{A}}_{\sigma,n}(\tau,\sigma) + \mathcal{B}_n\, f^{\mathcal{B}}_{\sigma,n}(\tau,\sigma)\right] \quad (4)$$

$$\mathbf{B}_\varphi(\tau,\sigma) = \mathcal{E}\, f^{\mathcal{E}}_{\varphi}(\tau,\sigma). \quad (5)$$

B. Determination of toroidal harmonic coefficients by fitting

Truncating the toroidal harmonic expansion of the magnetic flux density at $n = N$ yields the linear equation system

$$\begin{pmatrix}\mathbf{B}_\tau\\ \mathbf{B}_\sigma\\ \mathbf{B}_\phi\end{pmatrix} = \begin{pmatrix}0 & f^{\mathcal{A}}_\tau & f^{\mathcal{B}}_\tau\\ 0 & f^{\mathcal{A}}_\sigma & f^{\mathcal{B}}_\sigma\\ f^{\mathcal{E}}_\varphi & 0 & 0\end{pmatrix}\begin{pmatrix}\mathcal{E}\\ \mathcal{A}\\ \mathcal{B}\end{pmatrix}, \quad (6)$$

where the scalar functions and the toroidal harmonic coefficients are assembled in vectors, for example,

$$\mathcal{A}^t := (\mathcal{A}^t_0,\ldots,\mathcal{A}^t_N)^\top, \qquad f^{\mathcal{A}}_\tau := (f^{\mathcal{A}}_{\tau,0},\ldots,f^{\mathcal{A}}_{\tau,N}). \quad (7)$$

A finite number N of toroidal harmonic coefficients can be determined from observed magnetic flux density data by resolving the equation system (6) [1]. In the following, we refer to the matrix defined in (6) as *observation matrix* $\boldsymbol{H} \in \mathbb{R}^{3p\times 2(N+1)}$, where p is the number of measurement positions. Magnetic flux density observations in at least $p = N+1$ positions are required to obtain a full rank matrix $\boldsymbol{H}$ and a well-defined solution for the toroidal harmonic coefficients up to $n = N$. Since the observations are usually affected by uncertainties (e.g. measurement noise), we assume $p \geq N+1$ and seek the least squares solution of (6) by solving

$$\boldsymbol{H}^\top\boldsymbol{H}\begin{pmatrix}\mathcal{E} & \mathcal{A} & \mathcal{B}\end{pmatrix}^\top = \boldsymbol{H}^\top\begin{pmatrix}\mathbf{B}_\tau & \mathbf{B}_\sigma & \mathbf{B}_\phi\end{pmatrix}^\top \quad (8)$$

which is numerically implemented by LSQR [8].

C. Optimal sensor placement

The condition number κ_2 of the matrix $\boldsymbol{H}^\top\boldsymbol{H}$ is given by

$$\kappa_2(\boldsymbol{H}^\top\boldsymbol{H}) = \frac{\sigma_{\max}(\boldsymbol{H}^\top\boldsymbol{H})}{\sigma_{\min}(\boldsymbol{H}^\top\boldsymbol{H})} = \frac{\sigma_{\max}(\boldsymbol{H})^2}{\sigma_{\min}(\boldsymbol{H})^2}, \quad (9)$$

where $\sigma_{\max}, \sigma_{\min}$ are the maximal and minimal singular values. It is a measure of the sensitivity of the least squares solution to small data perturbations [9, theorem 3.5]. The smaller the condition number, the more stable the problem.

Since the coefficients of the observation matrix $\boldsymbol{H}$ depend on the sensor positions $\boldsymbol{\tau} := (\tau_1,\ldots,\tau_p)$ and $\boldsymbol{\sigma} := (\sigma_1,\ldots,\sigma_p)$, we seek to find an optimal experimental design by solving the following optimization problem

$$\min_{\boldsymbol{\tau},\boldsymbol{\sigma}\in\mathbb{R}^p} \quad \kappa_2(\boldsymbol{H}(\boldsymbol{\tau},\boldsymbol{\sigma})) \quad (10)$$

$$\text{s.t.} \quad \tau_{\text{out}} < \tau_i < 2.5\tau_{\text{out}} \quad \forall 1 \leq i \leq p$$
$$0 \leq \sigma_i \leq 2\pi \quad \forall 1 \leq i \leq p.$$

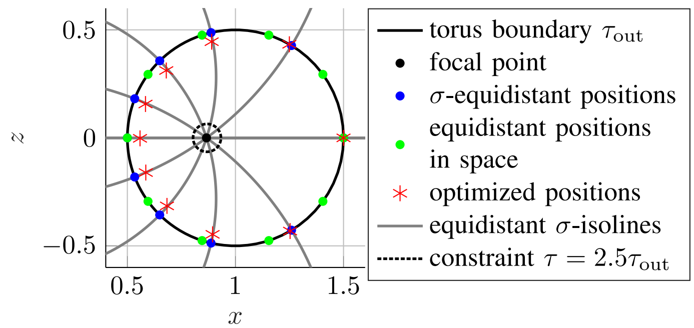

Figure 1. Sensor positions to determine the toroidal harmonic coefficients.

The optimization problem is solved using swarm optimization. To decrease the number of iterations, the search space is bounded by $\tau_i < 2.5\tau_{\text{out}}$ for all $1 \leq i \leq p$. Consequently, sensor positions everywhere in the cross-section of the torus besides a small disk around the focal point are admissible.

III. Application and Results

The optimization problem (10) is solved for a torus with aspect-ratio $\rho/R = 1/0.5$ and for $N = 9$ toroidal harmonic coefficients and $p = 10$ measurement positions and the normalization parameter $\tau_0 = \tau_{\text{out}}$ is chosen. Fig. 1 shows the sets of measurement positions with $\tau_i = \tau_{\text{out}}$ for all $1 \leq i \leq p$ and a σ-equidistant angular distribution (blue) and measurement positions with the same radius and angular equidistant distribution in space (green). The first measurement setup results in the condition number $\kappa_2(\boldsymbol{H}) = 3.33 \cdot 10^6$, and the second setup in $\kappa_2(\boldsymbol{H}) = 1.33 \cdot 10^7$. Thus, already small changes in the experimental design can improve the conditioning by one order of magnitude. The optimization results, shown in red in Fig. 1, have their angular distribution close to the σ-equidistant distribution and lead to the condition number $\kappa_2(\boldsymbol{H}) = 1.45 \cdot 10^6$.

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