Compatible Discrete Operators scheme to improve Yee's scheme for solving time domain Maxwell's equations

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***Abstract*—We present an approach to discretize Maxwell's equations called Compatible Discrete Operators (CDO). It is a low-order discretization that generalizes the classical FDTD scheme. In the CDO scheme the closure relations are discretized in a geometrically robust way that allows to deal with distorted and non-conformal polyhedral meshes. We briefly introduce the scheme, its link with the FDTD method and we give two numerical examples.**

I. INTRODUCTION

Geometrical accuracy, computational time and the size of the discrete problem are variables to optimize when dealing with simulations. The use of Cartesian meshes is a good way to increase the performances in terms of computational time and memory storage, but they are not well-adapted to capture geometrical details. The latter can be captured by using an unstructured mesh, leading to more complex numerical scheme. In this perspective, using polyhedral meshes seems to be a good trade-off between the accuracy and the computational cost. In this paradigm, there are two solutions. The first one, is to hybridize meshes, with unstructured and structured zones, and schemes [1], by assigning a scheme to each zone of the mesh and ensure the energy conservation at zones interfaces through time. The second approach, which will be considered in this paper, is to use schemes that are able to deal with hybrid meshes. The Compatible Discrete Operators (CDO) spatial discretization is one of these methods. It is a robust generalization of the finite difference method [2] that allows to deal with highly distorted and non-conformal polyhedral meshes. The presentation is done with the formalism proposed in [3], and lies on the work of [4], [2], [5].

II. CDO DISCRETIZATION

We recall Maxwell's equations in time-domain

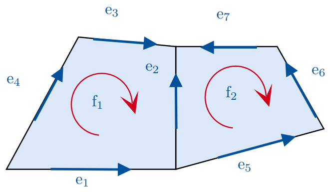
$$\partial_t \mathbf{B} - \nabla \times \mathbf{E} = 0, \tag{1}$$

$$\partial_t \mathbf{D} + \nabla \times \mathbf{H} = -\mathbf{J}, \tag{2}$$

$$\mathbf{D} = \varepsilon \mathbf{E}, \tag{3}$$

$$\mathbf{B} = \mu \mathbf{H}. \tag{4}$$

In the CDO discretization, each geometrical entity has a fixed orientation (Fig. 1) and the degrees of freedom are chosen according to the physical representation of the fields. Flux variables $\mathbf{D}$, $\mathbf{B}$ and $\mathbf{J}$ are attached to faces, and circulation variables $\mathbf{E}$ and $\mathbf{H}$ are attached to edges. The discretization lies on the decomposition of the equations into two parts: the topological law (1), (2) (independent of the material) and the constitutive equations (3), (4) (dependent on the material, possibly non-isotropic) [4].



$$\mathsf{CURL} = \begin{pmatrix} -1 & -1 & +1 & +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & -1 & -1 & -1 \end{pmatrix}$$

Figure 1: Example of a 2D mesh composed of 2 quadrangular cells and its incidence matrix $\mathsf{CURL} \in \mathbb{R}^{2\times 7}$ defined by the orientation of the faces and the edges.

A. Topological laws

The topological relations are discretized exactly using the Stokes theorem applied to circulations and fluxes (in three dimensions). It gives rise to discrete differential operators represented by incidence matrices [4], which verify the same physical properties as the continuous operators (for instance Poincaré's relations and adjunction properties). An example of CURL matrix, the discrete counter-part of the continuous $\nabla\times$ operator, for a 2D mesh is given in Fig. 1.

Discretized topological laws take the form:

$$\partial_t \underline{\mathsf{B}} - \mathsf{CURL}\ \underline{\mathsf{E}} = 0, \tag{5}$$

$$\partial_t \underline{\mathsf{D}} + \mathsf{CURL}^{\mathsf{T}} \underline{\mathsf{H}} = -\underline{\mathsf{J}}, \tag{6}$$

with $\underline{\mathsf{B}}, \underline{\mathsf{D}}, \underline{\mathsf{J}} \in \mathbb{R}^{N_\mathrm{f}}$, $\underline{\mathsf{E}}, \underline{\mathsf{H}} \in \mathbb{R}^{N_\mathrm{e}}$ and $\mathsf{CURL} \in \mathbb{R}^{N_\mathrm{f}\times N_\mathrm{e}}$. N_f and N_e are respectively the number of faces and the number of edges that compose the mesh.

B. Closure relations

The constitutive laws, represented by the Hodge operator $\star$ in differential geometry, are approximated by a linear operator whose algebraic representation is a symmetric positive definite matrix [2], [6]. Discretized closure relations take the form:

$$\underline{\mathsf{D}} = \mathsf{H}_\varepsilon \underline{\mathsf{E}}, \tag{7}$$

$$\underline{\mathsf{H}} = \mathsf{H}_{\mu^{-1}} \underline{\mathsf{B}}. \tag{8}$$

Different strategies can be applied to build these Hodge matrices, giving for each of them, a different scheme. For instance, if I_n is the identity matrix and if we consider a 3D orthogonal grid with uniform step Δx in each direction, the choice of Hodge matrices as:

$$\mathsf{H}_\varepsilon := \varepsilon \Delta x \mathrm{I}_n \quad \text{and} \quad \mathsf{H}_{\mu^{-1}} := \frac{1}{\mu \Delta x} \mathrm{I}_n,$$

leads exactly to the FDTD scheme. On polyhedral meshes they are defined [2] as:

$$(\mathsf{H}_\varepsilon)_{i,j} := \int_\Omega \varepsilon \ell_i(\mathbf{x}) \; \ell_i(\mathbf{x}) \; d\mathbf{x},$$
$$(\mathsf{H}_{\mu^{-1}})_{i,j} := \int_\Omega \frac{1}{\mu} h_i(\mathbf{x}) \; h_j(\mathbf{x}) \; d\mathbf{x},$$

where the basis functions ℓ_i and h_i are defined in order to take into account the geometrical properties of the polyhedral cells [7].

III. NUMERICAL EXAMPLES

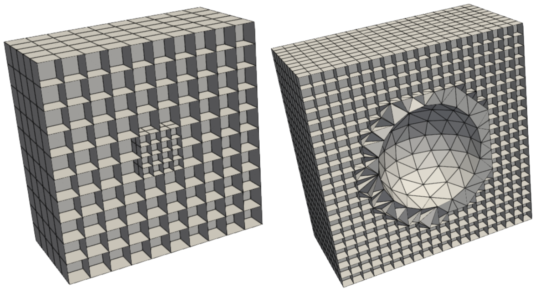


Figure 2: Meshes used in the examples.

The main feature of the CDO scheme is to deal with polyhedral meshes. This allows to make refinements and to mix the types of elements (see Figure 2).

A. *Mode propagation inside a cubic cavity*

The first numerical configuration (see Figure 2, left) is the comparison (see Figure 3) of the solution of the CDO scheme with an analytical solution in the case of the mode $(1, 1, 0)$ inside a locally refined cavity. The latter is chosen to be the unit cube $[0, 1]^3$.

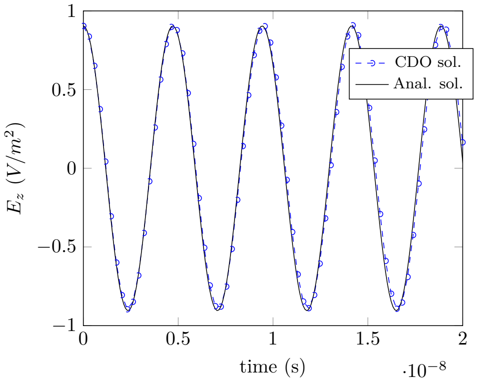


Figure 3: Solutions comparison: analytical (solid), CDO solution (dotted).

B. *Scatteting of a perfectly metallic sphere in free space*

The second configuration (see Figure 2 left) is the scattering of a plane wave on a perfectly metallic sphere Γ_s of radius $r = 0.25$ and center $O = (0.5, 0.5, 0.5)$. To simulate free-space, the computational domain is surrounded by Perfectly Matched Layers The incident field $\mathbf{E}_{\text{inc}}$ is given by a Gaussian plane wave:

$$\mathbf{E}_{\text{inc}} = \mathbf{E}_s e^{-\gamma^2},$$

with

$$\gamma = \frac{1}{\sigma}\big(t - t_0 - \sqrt{\varepsilon\mu}\; \mathbf{k} \cdot (\mathbf{x} - \mathbf{x}_0)\big), \; (\mathbf{x}, t) \in \Gamma_s \times \mathrm{I}.$$

The electric field amplitude $\mathbf{E}_s \in \mathbb{R}^3$ and the wave vector $\mathbf{k} \in \mathbb{R}^3$ are given by

$$\mathbf{E}_s = \begin{pmatrix}1 & 0 & 0\end{pmatrix}^T, \quad \mathbf{k} = \begin{pmatrix}0 & 0 & -1\end{pmatrix}^T,$$

while the wave starting position, $\mathbf{x}_0 \in \mathbb{R}^3$, the standard deviation, $\sigma > 0$, and the wave starting time, $t_0 > 0$, are

$$\mathbf{x}_0 = \begin{pmatrix}-0.1 & -0.1 & -0.1\end{pmatrix}^T, \quad \sigma = 9e^{-10}, \quad t_0 = 1e^{-8}.$$

In this example the CDO scheme is taken for different mesh refinement and is compared to the FDTD scheme with a spatial step of $1/160$. Figure 4 shows how geometry capturing the geometry helps the CDO scheme to improve the solution.

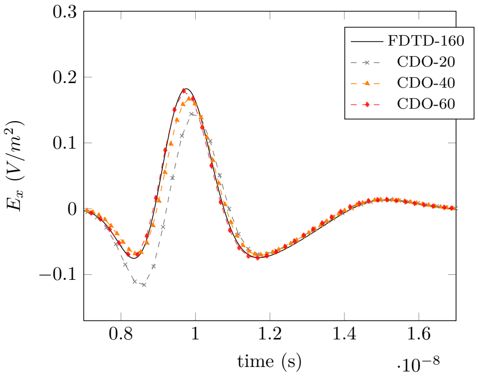


Figure 4: Scattering of a sphere with radius $r = 0.25m$.

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