Continuous Torque Formulation for Dual Magnetostatics using Shape Differentiation

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***Abstract*—This paper proposes a continuous formulation of the torque for the 2D non-linear dual magnetostatics based on the virtual work principle. Shape differentiation for a class of rigid body rotation of the rotor is performed, thus giving local forces and, ultimately, the torque. This process is detailed, and a comparison is proposed with the torque expression obtained from the primal form of 2D magnetostatics.**

I. Introduction

Torque computation approaches have been well-established from Maxwell's stress tensor methods [1], [2] with a relatively cheap computational cost to more accurate forms related to the shape differentiation of the coenergy [3], [4], [5]. Coulomb's [3] discrete derivation method effectiveness has been applied in many works in related applications such as electrical machines, on the contrary to continuous derivation methods [4], [5], due to the complexity of their implementation in finite element analysis. [4] bypasses the need for surface integrals by defining a small shell around the moving part of the system thus simplifying the implementation. To our knowledge, only [5] proposes a rigorous surface-integral implementation of the continuous shape differentiation of the coenergy for the primal form of magnetostatics. In this paper, we propose another expression of the torque. It is based on the differentiation of the energy for the dual-form magnetics. In that endeavor, we first pose the dual problem, determine the energy's shape derivative, and then choose a morphism representing the rigid-body rotation of the rotor to derive the torque.

II. Formulation and derivation

A. Problem formulation

To derive the torque equation, we use the following simplified model Fig.1 where Ω_r is the rotor domain, Ω_g is akin to a slice of the airgap. Ω_i and Ω_a are, respectively, the ferromagnetic and air domains, and Γ is their interface. A magnetic source term $\beta(x,y)$ is imposed only on the outer boundary $\partial\Omega_e$. We assume that only two materials are in the rotor: the air and a ferromagnetic isotropic material with non-linear and anhysteretic magnetic properties represented by a permeability μ. For this type of geometry, we can express the strong form of the 2D magnetostatics dual problem as such

$$\begin{cases} -\nabla.(\mu(|\nabla\phi_i|)\nabla\phi_i) = 0 & x \in \Omega_i \\ -\nabla.(\mu_0\nabla\phi_a) = 0 & x \in \Omega_a \\ \mu_0\nabla\phi_a.n = \beta(x,y).n & x \in \partial(\Omega_g \cup \Omega_r) \\ \phi_i = \phi_a & x \in \Gamma \\ \mu(|\nabla\phi_i|)\nabla\phi_i.n = \mu_0\phi_a.n & x \in \Gamma \end{cases} , \quad (1)$$

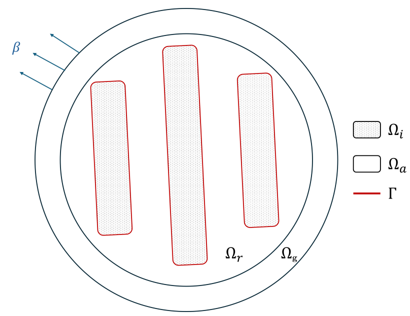

Figure 1. Simplified geometry of an electrical machine

where ϕ is the magnetic scalar potential (i.e $h = -\nabla\phi$), ϕ_a and ϕ_i are the solutions to the direct problem, and μ_0 is the vacuum permeability. For the sake of clarity, we define the following material law operator χ (2) such that $b = -\chi(\nabla\phi)$ along with its Jacobian $D\chi$.

$$\chi(h) := \mu(|h|)h \; ; \; D\chi(h) := \mu(|h|)I + \frac{\mu'(|h|)}{|h|} h \otimes h \quad (2)$$

The magnetic energy of the system is defined by (3).

$$W_m = \int_{\Omega_a} \frac{\mu_0}{2}|\nabla\phi_a|^2 \,\mathrm{d}x + \int_{\Omega_i} (\chi(\nabla\phi_i).\nabla\phi_i - \int_0^{|\nabla\phi_i|} |\chi(h)| \,\mathrm{d}|h|) \,\mathrm{d}x \quad (3)$$

B. Lagrangian formulation

To obtain the shape derivative of the energy $W_m(\Omega,\phi)$, we search for a saddle point of the associated Lagrangian $\mathcal{L}$ defined in (4). We are using the adjoint variable method to infer a descent direction.

$$\begin{aligned} \mathcal{L} = &\int_{\Omega_a} \frac{\mu_0}{2}|\nabla\phi_a|^2 \,\mathrm{d}x + \int_{\Omega_i} (\chi(\nabla\phi_i).\nabla\phi_i - \int_0^{\nabla\phi_i} |\chi(h)| \,\mathrm{d}|h|) \,\mathrm{d}x \\ &+ \int_{\Omega_a} -\nabla.(\mu_0\nabla\phi_a)p_a \,\mathrm{d}x + \int_{\Omega_i} -\nabla.(\chi(\nabla\phi_i))p_i \,\mathrm{d}x \\ &+ \int_\Gamma \lambda\,(\phi_a - \phi_i) \,\mathrm{d}x + \int_\Gamma \mu\,(\mu_0\nabla\phi_a.n - \chi(\nabla\phi_i).n) \,\mathrm{d}x \\ &+ \int_{\partial(\Omega_g\cup\Omega_r)} \eta(\mu_0\nabla\phi_a.n - \beta.n) \,\mathrm{d}x \end{aligned} \quad (4)$$

Thus, we need to determine $\phi_a, \phi_i, p_a, p_i, \lambda, \mu, \eta$ such that the equality between the shape derivative of the energy and the one of the Lagrangian (5) is verified. Here, V is a vector displacement field of class C^1. This equality is equivalent to cancel each partial derivative of $\mathcal{L}$, except the one with regard to $\Omega = \Omega_i \cup \Omega_a$.

$$\mathcal{D}W_m(\Omega)(V) = \mathcal{D}\mathcal{L}(\phi_a, \phi_i, p_a, p_i, \lambda, \mu, \eta)(V) \qquad (5)$$

By definition of the Lagrangian, from $\partial_{p_a}\mathcal{L}(v) = 0$ and $\partial_{p_i}\mathcal{L}(v) = 0$, we deduce the state equations in the bulk Ω_a and Ω_i (1). Similarly, from the partial derivative $\partial_\lambda, \partial_\mu, \partial_\eta$, the transmission conditions at the material interface Γ for the potential ϕ are deducted. The remaining partial derivatives are more troublesome; we choose to detail ∂_{ϕ_i} below.

$$\begin{aligned}\partial_{\phi_i}\mathcal{L}(v) = &\int_{\Omega_i} (D\chi(\nabla\phi_i)\nabla\phi_i.\nabla v - \nabla.((D\chi(\nabla\phi_i)\nabla v)p_i \,\mathrm{d}x \\ &- \int_\Gamma (\lambda v + \mu D\chi(\nabla\phi_i)\nabla v.n) \,\mathrm{d}x = 0\end{aligned} \qquad (6)$$

To properly derive the strong form of the adjoint equation, the divergence theorem is used twice, thus (6) becomes (7).

$$\begin{aligned}\partial_{\phi_i}\mathcal{L}(v) = &\int_{\Omega_i} -\nabla.(D\chi(\nabla\phi_i)\nabla(\phi_i + p_i))v \,\mathrm{d}x \\ &- \int_\Gamma ((\lambda - D\chi(\nabla\phi_i)(\nabla p_i + \nabla\phi_i).n)v \,\mathrm{d}x \\ &- \int_\Gamma (\mu + p_i) D\chi(\nabla\phi_i)\nabla v.n \,\mathrm{d}x = 0\end{aligned} \qquad (7)$$

Assuming that $D\chi(\nabla\phi_i)$ has strictly positive eigenvalues - which is equivalent to (1) being well-posed-, one can derive the adjoint equation in the bulk and on Γ choosing sequentially v with an arbitrary trace on Γ and $D\chi(\nabla\phi_i)\nabla v.n = 0$ and the converse. Thus, we get (8):

$$\begin{cases} p_i = -\phi_i & x \in \Omega_i \\ \mu = -p_i & x \in \Gamma \\ \lambda = D\chi(\nabla\phi_i)(\nabla p_i + \nabla\phi_i).n & x \in \Gamma \end{cases} \qquad (8)$$

With similar developments, the adjoint problem can be determined in Ω_a, and thus the full on the whole domain $p = -\phi$. (5) can hence be verified. For a detailed explanation in the linear case see [6, section 4.3]

C. Shape differentiation

As ϕ satisfies (1) we get, after simplifying the expression:

$$\mathcal{D}\mathcal{L}(V) = \int_\Gamma ([w_m^{(k)}(\nabla\phi_k)]_a^i + (\mu_0 - \mu(|\nabla\phi_i|))|\nabla_T\phi|^2)V.n \,\mathrm{d}x \qquad (9)$$

where $w_m^{(k)}$ is the density of the energy in Ω_k and ∇_T is the tangential gradient. To get the expression of the torque, we apply to the domain a smooth morphing $\mathcal{R}$, akin to a rotation of angle θ in Ω_r:

$$\mathcal{R}(x,\theta) = \begin{cases} \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & x \in \Omega_r \\ 0 & x \in \partial\Omega_e \end{cases} \qquad (10)$$

Using the Taylor development at $\theta = 0$ we can identify V_r:

$$\begin{aligned}\mathcal{R}(x,\theta) &= x + \theta\,\partial_\theta\mathcal{R}(x,0) + \mathcal{O}(\theta^2) \\ &= x + \theta\,V_r + \mathcal{O}(\theta^2)\end{aligned} \qquad (11)$$

Recall the formula of the torque from the virtual work principle, for this peculiar V_r

$$T(\phi) = -\frac{\partial W_m}{\partial\theta} = -\mathcal{D}\mathcal{L}(V_r) \qquad (12)$$

Thus, we get the expression of the torque for 2D non-linear dual magnetostatics.

$$\begin{aligned}T(\phi) = &-\int_\Gamma [w_m^{(k)}(\nabla\phi_k)]_a^i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} .n \,\mathrm{d}x \\ &- \int_\Gamma (\mu_0 - \mu(|\nabla\phi_i|))|\nabla_T\phi|^2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} .n \,\mathrm{d}x\end{aligned} \qquad (13)$$

D. Primal - dual equivalences

For $B = \begin{pmatrix} \partial_{x_2}u & -\partial_{x_1}u \end{pmatrix}^T$, we have the equalities:

$$\nu(|\nabla u|)\nabla u.n = -\nabla\phi.t \quad , \quad \nabla u.n = -\chi(\nabla\phi).t \qquad (14)$$

where u is the x_3-component of magnetic vector potential. Using the Legendre transformation on $w_m^{(k)}$ and (14), we get

$$\begin{aligned}T(u) = &\int_\Gamma [w_c^{(k)}(\nabla u_k)]_a^i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} .n \,\mathrm{d}x \\ &+ \int_\Gamma (\nu_0 - \nu(|\nabla u_i|))|\nabla_T u|^2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} .n \,\mathrm{d}x\end{aligned} \qquad (15)$$

where $w_c^{(k)}$ is the density of coenergy in the domain Ω_k. We get back the formula (15) found in [5] and thus the well-known expression of the torque (16).

$$T = \frac{\partial W_c}{\partial\theta}(u) = -\frac{\partial W_m}{\partial\theta}(\phi) \qquad (16)$$

III. Conclusion

This paper proposes a continuous formula for the torque as the shape derivative of the magnetic energy for dual magnetostatics. A rigorous analysis, searching for the saddle point of a Lagrangian operator was done to get the shape derivative. A morphism akin to a rotation movement of the rotor was then chosen to get the torque expression. Finally, the expression found was equivalent to the torque by coenergy derivation formula for primal magnetostatics.

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