Integral equation numerical approach for the determination of the diffraction coefficients from generally shaped cones

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***Résumé*—Knowledge of the diffraction coefficients of canonical problems is a fundamental component for analyzing wave propagation in complex environments using ray-based algorithms. These latter are commonly used in propagation channel description (as 5G applications), to design and optimize antennas in realistic environments, lenses, and other systems that involve wave propagation. In this paper, we present a numerical method for the calculation of the diffraction coefficient for cones of arbitrary transverse sections.**

***Keywords*—electromagnetic diffraction and scattering, high-frequencies, integral equations**

I. INTRODUCTION

The electromagnetic diffraction of waves from cones with arbitrary sections is one of the so-called "canonical scattering problems", which constitute ideal objects of infinite dimension with a distinct geometric feature (tip, edge) that could be solved wholly analytically or piece-wise analytically and numerically [1]. The diffraction coefficients are expressed by separating the problem into two Sturm Liouville problems with variables the 2-dimensional transverse variable $\hat{\mathbf{r}}$ and the 1-dimensional radial variable r. Analytical solutions for the diffraction by circular and elliptic cones have been presented and validated [2]. In the case of an arbitrary section cone, the solution of the transverse differential equation cannot be derived analytically, as the arbitrary section of the boundary curve of the cone comes up with an in-homogeneity and therefore a numerical approach is applied [3] [4], the Method of Moments solution.

The numerical integration of the diffraction integral can then be obtained, having ensured an accurate calculation of the transverse function by comparing it with other analytical and numerical solutions. The solution of this integral, derived by Smyslyaev [5], is generally considered a challenging task. The last claim is relevant as the integrand can be slowly convergent or even divergent at the non singular-reflected directions. Finally, the methods and algorithms, referred in [6] will be applied to force this integral to converge rapidly.

II. THE DIFFRACTION PROBLEM

The far-field diffraction from an arbitrarily shaped cone is examined. Figure 1 shows the general geometry. The observation and source point are indicated by the vector points $\mathbf{r}$ and $\mathbf{r}'$, respectively. The cone in our analysis stretches to infinity. The cone section is described by the curve ∂S defined as the intersection between the cone and the unit sphere centered at the tip of the cone.

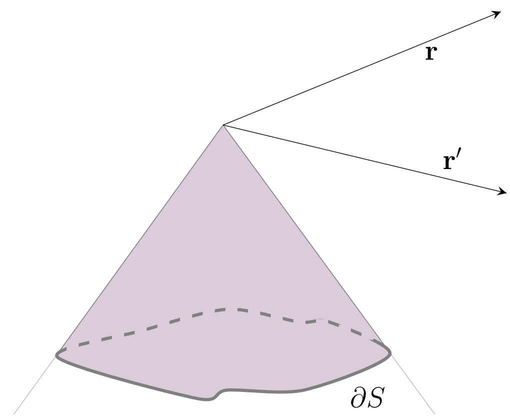


FIGURE 1 – The diffraction problem

Following the approach presented in [3], by introducing a spherical reference system with the origin at the cone tip and separating the radial coordinate from the angular coordinates, the diffraction coefficient (i.e., the cone scattered field in the plane-wave—far field regime) is cast in the Watson integral representation form [7]

$$D\left(\hat{\mathbf{r}}, \hat{\mathbf{r}}'\right) = \frac{1}{k} \oint_C g_t\left(\hat{\mathbf{r}}, \hat{\mathbf{r}}'; \lambda\right) \sin\left(\sqrt{\lambda + \tfrac{1}{4}}\pi\right) d\lambda \qquad (1)$$

with k the wave number, λ the eigenvalues of the transverse problem and C any closet path on the complex plane of λ that encloses the signularities of g_t and abstain from the brunch cut of the 1-dimensional radial Sturm Liouville differential equation [5].

The transverse Green's function g_t is the solution of the wave equation 2D problem on the unit sphere

$$\begin{cases} \left[r^2 \nabla_t \cdot {}_t\nabla + \lambda\right] g_t\left(\hat{\mathbf{r}}, \hat{\mathbf{r}}'; \lambda\right) = -\delta\left(\hat{\mathbf{r}} - \hat{\mathbf{r}}'\right), & \text{in } S, \\ g_t\left(\hat{\mathbf{r}}, \hat{\mathbf{r}}'; \lambda\right) = 0 \text{ or } \dfrac{\partial}{\partial n} g_t\left(\hat{\mathbf{r}}, \hat{\mathbf{r}}'; \lambda\right) = 0, & \text{on } \partial S \end{cases} \qquad (2)$$

with S denoting the part of the unit sphere external to the cone. Either Dirichlet or Neumann boundary conditions are imposed on the cone section ∂S. The transverse Beltrami-Laplace operator on the unit sphere in spherical coordinates reads [8]

$$\nabla_t \cdot {}_t\nabla = \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \quad (3)$$

When the cone section can be mapped on suitable coordinates, the transverse Green's function can be calculated analytically with the separation of variables [2] ; however, in the more general case of a cone with an arbitrarily shaped section, one can resort to a numerical approach.

III. NUMERICAL SOLUTION OF THE TRANSVERSE FUNCTION

An integral equation for g_t can be obtained by using the transverse free space Green's function g_t^0 and the 2nd Green's theorem identity.

$$\begin{aligned} g_t\left(\hat{\mathbf{r}}, \hat{\mathbf{r}}'; \lambda\right) &= g_t^0\left(\hat{\mathbf{r}}, \hat{\mathbf{r}}'; \lambda\right) \\ &\quad - \oint_{\partial S}\left[g_t\left(\hat{\mathbf{s}}, \hat{\mathbf{r}}'; \lambda\right)\frac{\partial}{\partial n_s}g_t^0\left(\hat{\mathbf{r}}, \hat{\mathbf{s}}; \lambda\right)\right. \\ &\quad \left. - g_t^0(\hat{\mathbf{r}}, \hat{\mathbf{s}}; \lambda)\frac{\partial}{\partial n_s}g_t\left(\hat{\mathbf{s}}, \hat{\mathbf{r}}'; \lambda\right)\right]d\gamma_S. \end{aligned} \quad (4)$$

where $\frac{\partial}{\partial n_s}$ is the angular derivative in the direction normal to S, $\hat{\mathbf{s}}$ is the direction on the unit sphere pointing at the integration points along the path ∂S, parametrized by the angle γ_S. A numerical solution of the previous equation can be performed using standard numerical techniques for integral equations as the Method of Moments (MoM).

TABLE I – The integral equations for each case, soft and hard.

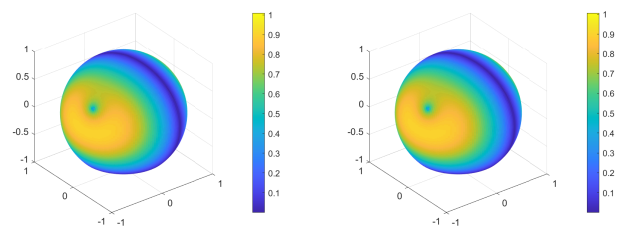
g_t	Soft	Hard 1st	Hard 2nd
Closet boundary	✓	✓	
Open boundary	✓	✗	✓

Then, the soft and hard case boundary conditions are imposed and two different integral equations are thus obtained. For the soft case, a 1st kind of Fredholm integral equation is occured, thus both close and open path boundaries can be solved. However, for the hard case, a 2nd kind Fredholm equation is obtained leading to the impossibility of analyzing open path boundary problems because of the discontinuity of the integral kernel function at the boundary. For this reason, by deriving the integral equation with respect to the normal direction of the observation point, a second integral equation of the 1st kind can be attained to overcome the previously mentioned issues. Table I summarizes the properties of the three considered integral equations.

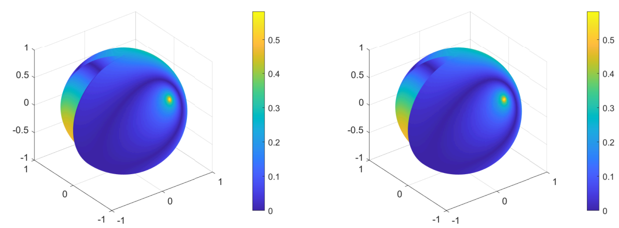
TABLE II – The type of basis-test functions

g_t	Soft	Hard 1st	Hard 2nd
Unit pulses	✓	✓	✗
Triangular functions			✓

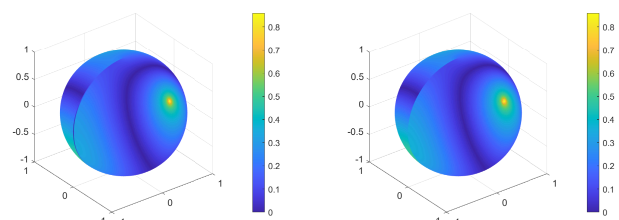
The integral equations are solved by using the MoM Galerkin technique. The boundary ∂S (belonging to the unit sphere) is divided into curved segments (mesh) using a curvilinear parametrization, while two types of basis-test functions have been used to describe the solution : the unit pulses and the triangular functions. This latter choice has been made to simplify as much as possible the formulation and minimize the computational effort. For both the soft and hard case 1st kind integral equation, the unit pulses basis functions were sufficient to regularize the MoM impedance self-reaction terms by analytically extracting the singularities that the kernel function creates. However, for the hard case 2nd kind integral equation, div-conforming basis function as the triangular one must be used to extract the kernel singularities [Table II].



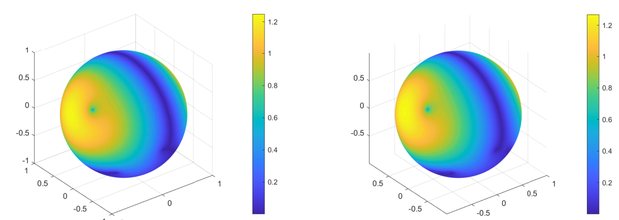
a) circular cone, aperture angle $\pi/4$, soft case.



b) wedge, aperture angle $\pi/4$,
hard case 1st integral equation.



c) half plane, hard case 2nd integral equation.



d) piecewise elliptic cone, $a = 1$, $b = 5$, with 4 shells, aperture angles $\theta_i = \{i\pi/2, (2i-1)\pi/4\}$, $i = 1, 2, 3, 4$, soft case [3].

FIGURE 2 – The $|g_t|$ in 3D space of M.o.M. (left) and analytical solution and finite differences (right). It is choosen an arbitrary source point $\theta = \pi/3$ and $\phi = 3\pi/5$, $\lambda = 3.1$ and number of segments 700.

IV. RESULTS

The above numerical procedure has been validated by comparison with analytical solutions available in the literature for simple geometries and a finite differences method for more complicated ones. Numerical results for a circular cone, a wedge, a half-plane, and a piecewise elliptic cone are shown in Figure 2. All the geometrical parameters of the problems are described in the caption. The obtained numerical results are in close agreement with the reference ones.

V. CONCLUSION

In conclusion, a final procedure is provided to accurately find the transverse function, which is included in the integral describing the diffraction coefficients of a general cone. Further steps in this research will involve numerical computation, enforcing convergence, and speeding up this integral.

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