On analytic formulas useful to solve the inverse magnetization problem

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***Abstract*—This work takes its roots in the inverse magnetization problem, where the magnetization properties of rocks are sought from measurements of the magnetic field they generate, taken on a horizontal plane at some height above the rocks. The Poisson kernel being at the core of the corresponding magnetic potential, double integrals of functions over the cube of a distance are ubiquitous in this context. We present explicit formulas for the particular case when the numerator of the integrand is a polynomial of two variables and show how these formulas can be useful to address the inverse magnetization problem.**

I. Introduction

The general problem motivating this work is the problem of recovering some characteristics of the magnetization of a rock specimen from measurements of the generated magnetic field taken on a horizontal plane above it. This setup is typical (though, of course, with very different scales), *e.g.*, of measurements obtained from a flight over a region presenting a magnetic anomaly [8] or from measurements of the magnetic field of a small rock sample obtained from a scanning magnetic microscope [1]. We model this setup as in Fig. 1: a magnetic sample, enclosed in a bounded volume $\mathcal{A}$, carries a magnetization density $\boldsymbol{m} = (m_1, m_2, m_3)$ which produces a magnetic field $\boldsymbol{B}$ measured at points $\boldsymbol{x} = (x_1, x_2, x_3)$ of a square Q at some fixed distance above. From Maxwell's equations in the static case [6, Sec. 5.9.C], $\boldsymbol{B} = -\mu_0 \nabla\phi$ where the scalar magnetic potential ϕ is given by

$$\phi(\boldsymbol{x}) = \frac{1}{4\pi} \iiint_{\mathcal{A}} \frac{\langle \boldsymbol{m}(\boldsymbol{t}), \boldsymbol{x} - \boldsymbol{t}\rangle}{\|\boldsymbol{x} - \boldsymbol{t}\|^3} \, \mathrm{d}\boldsymbol{t}\,, \tag{1}$$

where $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = x_1y_1 + x_2y_2 + x_3y_3$ denotes the usual scalar product and $\|\boldsymbol{x}\| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$.

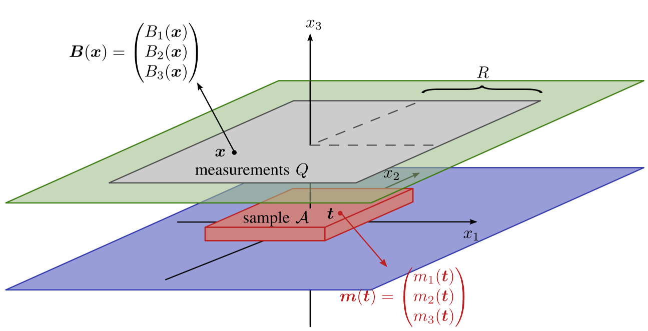

Figure 1. Schematic setup with notations

The full recovery of $\boldsymbol{m}(\boldsymbol{t})$ for all $\boldsymbol{t} \in \mathcal{A}$ from the observed data is an ill-posed problem (it does not even have a unique solution [2]) but an interesting characteristic of the magnetization, its net moment $\langle \boldsymbol{m}\rangle = \iiint_{\mathcal{A}} \boldsymbol{m}(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t}$, is well defined from the data (though it remains unstable with respect to noise and calls for regularization techniques [1]). For this purpose, an old idea (see, *e.g.*, [4]) is to integrate a component B_j of $\boldsymbol{B}$ (where $j \in \{1,2,3\}$) against simple functions $u(x_1, x_2)$. When $u(x_1, x_2)$ is a polynomial of small degree, and because he popularized them, these are called Helbig's integrals [5]. Accounting for the fact that B_j is only known on Q we introduce the direct operator B_j which maps $\boldsymbol{m} \in [L^2(\mathcal{A})]^3$ to the corresponding component of the field, seen as a function[1] of $L^2(Q)$. Then, by definition, its adjoint operator $\boldsymbol{B_j}^\star : L^2(Q) \to [L^2(\mathcal{A})]^3$ satisfies

$$\iint_Q u(x_1,x_2) B_j[\boldsymbol{m}](\boldsymbol{x})\,\mathrm{d}x_1\mathrm{d}x_2 = \iiint_{\mathcal{A}} \langle \boldsymbol{B_j}^\star[u](\boldsymbol{t}), \boldsymbol{m}(\boldsymbol{t})\rangle\,\mathrm{d}\boldsymbol{t}.$$

Any piece of information on the structure of $\boldsymbol{B_j}^\star[u]$ could hence be useful to understand the effect of integrating u against magnetic field data.

II. Results

First, we observe that the i-th component $\boldsymbol{B_j}^\star[u]_{|i}$ of $\boldsymbol{B_j}^\star[u]$ (where $i \in \{1,2,3\}$) can be made more explicit. Replacing $B_j[\boldsymbol{m}](\boldsymbol{x})$ in the above equation by $\partial_{x_j}\phi$ where ϕ is given by Eq. (1), then using Fubini's theorem to swap the integrals with respect to $\boldsymbol{x}$ and $\boldsymbol{t}$ and considering only the part in factor of m_i, we get a first expression for $\boldsymbol{B_j}^\star[u]_{|i}$, from which, doing the change of variable $\boldsymbol{y} = \boldsymbol{t} - \boldsymbol{x}$, we get

$$\boldsymbol{B_j}^\star[u]_{|i}(\boldsymbol{t}) = \frac{-\mu_0}{4\pi} \iint_{Q_{\boldsymbol{t}}} (u \circ s_{\boldsymbol{t}})(y_1,y_2)\,\partial_{y_j}\left[\frac{y_i}{\|\boldsymbol{y}\|^3}\right] \mathrm{d}y_1\mathrm{d}y_2,$$

where $Q_{\boldsymbol{t}} = [t_1 - R, t_1 + R] \times [t_2 - R, t_2 + R]$ and $s_{\boldsymbol{t}}(y_1, y_2) = (t_1 - y_1, t_2 - y_2)$.

Now, observing that $\partial_{y_j}(y_i/\|\boldsymbol{y}\|^3) = \partial_{y_i}(y_j/\|\boldsymbol{y}\|^3)$ and $\sum_{k=1}^3 \partial_{y_k}(y_k/\|\boldsymbol{y}\|^3)$, we get that the matrix $(\boldsymbol{B_j}^\star[u]_{|i})_{i,j}$ is symmetric with a vanishing trace. We hence further suppose without loss of generality that $j \in \{1,2\}$.

After an integration by part, one is reduced to integrate (wrt. a single variable) $y_i(u \circ s_{\boldsymbol{t}})(y_1,y_2)/\|\boldsymbol{y}\|^3$ and integrate (wrt. both variables) $y_i[\partial_{y_j}(u \circ s_{\boldsymbol{t}})(y_1,y_2)]/\|\boldsymbol{y}\|^3$. The case when $u(y_1, y_2)$ is a polynomial is of particular interest: the problem boils down to integrating (once or twice) a polynomial over $\|\boldsymbol{y}\|^3$. For small degrees [7, Ch. 7] or specific cases [9, Ch. 1.2.43], explicit formulas are available in textbooks, but as far as we know, the following formulas, expressing the

[1] By abuse of notation, we identify here $Q = [-R, R]^2 \times \{x_3\}$ (where x_3 is a fixed value) which is, really, a subset of $\mathbb{R}^3$, with $[-R, R]^2 \subset \mathbb{R}^2$.

double integral with a simple recurrence on the coefficients of the polynomial u, do not exist in literature.

Proposition 1. If $u(x_1,x_2)$ is a generic polynomial $u(x_1,x_2)=\sum_{0\le k+m\le p}\alpha_{k,m}\,x_1^k x_2^m$, then

$$\begin{aligned}\iint \frac{u(x_1,x_2)}{\|\boldsymbol{x}\|^3}\,\mathrm{d}x_1\mathrm{d}x_2 &= a\,A(\boldsymbol{x})\\ &+\left(\sum_{k=0}^{p-1} u_k\,x_1^k\right)L_2(\boldsymbol{x})+\left(\sum_{m=0}^{p-1} v_m\,x_2^m\right)L_1(\boldsymbol{x})\\ &+\left(\sum_{0\le k+m\le p-2} w_{k,m}\,x_1^k x_2^m\right)\|\boldsymbol{x}\|, \qquad (2)\end{aligned}$$

where $L_i(\boldsymbol{x})=\operatorname{argtanh}(x_i/\|\boldsymbol{x}\|)$ $(i\in\{1,2\})$, $A(\boldsymbol{x})=\arctan(x_1x_2/(x_3\|\boldsymbol{x}\|))$ and $\{(w_{k,m}),(u_k),(v_m),a\}$ are real numbers.

Sketch of the proof. Differentiating both sides of Eq. (2) with respect to x_1 and x_2, then multiplying by $\|\boldsymbol{x}\|^3$, we get[2]:

$$\begin{aligned}u(x_1,x_2)&=\sum_{2\le k+m\le p} W_{k,m}x_1^k x_2^m\\ &+x_2^2\sum_{k=1}^{p+2}(k+1)U_{k+1}x_1^k+x_1^2\sum_{m=1}^{p+2}(m+1)V_{m+1}x_2^m\\ &+\sum_{k=1}^{p+2}U_k\,x_1^k+v_2x_1^2+\sum_{k=1}^{p+2}V_k\,x_2^k+u_2x_2^2\\ &+ax_3+x_3^2v_1+x_3^2u_1+x_3^4w_{1,1}. \qquad (3)\end{aligned}$$

where,

$$\begin{aligned}W_{k,m}=&f_{k,m}(w_{k-1,m-1},w_{k-3,m+1},w_{k+1,m-3},w_{k+1,m-1}\\ &w_{k-1,m+1},w_{k+1,m+1}),\\ U_k=&g_k(u_{k+1},u_{k-1},w_{k-3,1},w_{k-1,1},w_{k+1,1}),\\ V_k=&h_k(v_{k+1},v_{k-1},w_{1,k-3},w_{1,k-1},w_{1,k+1}). \qquad (4)\end{aligned}$$

such that $(f_{k,m})$, (g_k) and (h_k) are known $\mathbb{R}$-valued linear functions. Equating corresponding monomials in Equation (3) yields a square linear system of equations with $\{(w_{k,m}),(u_k),(v_m),a\}$ as the unknowns. Moreover, due to the specific structure of Eq. (4), an efficient recursive scheme can be implemented to solve this system instead of directly inverting the coefficient matrix. Namely, starting from $n=p-2$ and decreasing until $n=1$, we are led at each step to the resolution of two tridiagonal systems to get the vector $z_n=(v_{n+1},w_{0,n},..w_{k,n-k},..,w_{n,0},u_{n+1})$ (where $z_n=0$ for $n>p-2$). On the other hand, $z_0=(v_1,w_{0,0},u_1)$, $z_{-1}=(v_0,u_0)$ and $z_{-2}=a$ are obtained via straightforward computations. To illustrate, if n is even, we obtain a system for the odd-indexed unknowns:

$$\begin{pmatrix} b_1 & c_1 & 0 & \dots & 0\\ a_3 & b_3 & c_3 & \dots & 0\\ 0 & a_5 & b_5 & \dots & 0\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & \dots & a_{n+3} & b_{n+3}\end{pmatrix}\begin{pmatrix} v_{n+1}\\ \vdots\\ w_{\mathrm{odd,odd}}\\ \vdots\\ u_{n+1}\end{pmatrix}=\begin{pmatrix} d_1\\ \vdots\\ d_{\mathrm{odd}}\\ \vdots\\ d_{n+3}\end{pmatrix}$$

where the (a_k), (b_k) and (c_k) are known integers, and the (d_k)'s are functions of z_{n+2}, z_{n+4} and the $(\alpha_{k,m})$. A similar system permits retrieving the even-indexed parameters.

III. Illustration on an Example

In order to illustrate a potential use of the proposed formulas, we consider an example inspired from [1]. Therein, a method is proposed to compute good linear estimators for the total net moment of a rock sample. At some point, the method involves evaluating $\boldsymbol{B_3}^\star[u](\boldsymbol{t})$, for $\boldsymbol{t}\in\mathcal{A}=[-s,s]^2\times\{0\}$, where u is an eigenfunction of the Laplacian on the square, which involves (among other computations) evaluating

$$\int_{t_1-R}^{t_1+R}\int_{t_2-R}^{t_2+R} f(x_1)g(x_2)\frac{1}{\|\boldsymbol{x}\|^3}\,\mathrm{d}x_1\mathrm{d}x_2,$$

where $f(x_1)=\cos\left(\frac{n_1\pi x_1}{2R}\right)$ and $g(x_2)=\sin\left(\frac{n_2\pi x_2}{2R}\right)$ with n_1 and n_2 two integers.

For the sake of the example, say that $n_1=6$ and $n_2=4$, $R=2.55\mathrm{e}{-3}$, $s=1.97\mathrm{e}{-3}$ and $x_3=0.27\mathrm{e}{-3}$. Using the dedicated Sollya tool [3] we can compute polynomials $p_i(x_i)$ $(i=1,2)$ with binary64 coefficients such that $|p_1(x_1)-f(x_1)|\le 1\mathrm{e}{-5}$, $|p_2(x_2)-g(x_2)|\le 1\mathrm{e}{-5}$ for all $x_i\in[-s-R,s+R]$. To achieve it, we choose p_1 even and of degree 28, while p_2 is odd and of degree 21. Using the recurrence formulas described in the previous section, we explicitly compute an indefinite double integral of $u(x_1,x_2)/\|\boldsymbol{x}\|^3$ where $u(x_1,x_2)=p_1(x_1)p_2(x_2)$, from which the value of the integral is deduced by evaluation of that double primitive at the bounds.

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[2]We conventionally assume that parameters with indices exceeding the predefined limits are equal to zero.