A robust and adaptive GenEO-type domain decomposition preconditioner for $\mathrm{H}(\mathrm{curl})$ problems in non simply connected three-dimensional geometries

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***Abstract*—We develop and analyse domain decomposition methods for linear systems of equations arising from conforming finite element discretisations of positive Maxwell-type equations, namely for $\mathbf{H}(\mathbf{curl})$ problems. Convergence of domain decomposition methods rely heavily on the efficiency of the coarse space used in the second level. We design adaptive coarse spaces that complement a near-kernel space made from the gradient of scalar functions. The new class of preconditioner is based on spectral coarse spaces, and is specially designed for curl-conforming discretisations of Maxwell's equations in non simply connected geometries (i.e. not all loops are reducible to a point).**

I. Introduction

In this work we focus on the efficient solution of the following Maxwell problem:

$$\begin{aligned} \nabla\times(\mu^{-1}\nabla\times\boldsymbol{E})+\gamma\varepsilon\boldsymbol{E}=\boldsymbol{f} \quad &\text{in } \Omega,\\ \boldsymbol{E}\times\boldsymbol{n}=0 \quad &\text{on } \partial\Omega. \end{aligned} \tag{1}$$

Here $\boldsymbol{E}$ is the vector-valued electric field, $\boldsymbol{f}$ is a source term, while μ and ε are electromagnetic parameters which are uniformly bounded and strictly positive but which we allow to be heterogeneous. Further, $\gamma > 0$ is a positive parameter which is allowed to be very small. We suppose Ω is a computational domain (possibly not simply connected) and $\boldsymbol{n}$ is the outward normal to $\partial\Omega$. This problem, although positive definite, remains challenging from the solution methods point of view. The reference method to solve this problem, remains up to now, the celebrated algorithm of Hiptmair–Xu from [4], which was identified by the U.S. Department of Energy in 2008 as one of the top ten recent breakthroughs in computational science.[1] For a trivial topology, this algorithm is robust w.r.t. mesh size. But theory and practice (see Table I) show this is not the case for non simply connected domains. This is in contrast with the domain decomposition method we use here which is provably robust for non trivial topologies.

II. Coarse Space of the two-level Schwarz method

Let A denote the discretization matrix of (1) for a Nedelec finite element. We consider a domain decomposition into N subdomains and denote by R_i the restriction of a global vector of d.o.f.'s to its local component in $\mathcal{N}_i$ of subdomain i and D_i the partition of unity matrix associated to it, see [1].

[1]Report of The Panel on Recent Significant Advancements in Computational Science.

Definition 1 (Split near-kernel coarse space): Let G be the easily computable part of the near-kernel of A; namely, in our case, vectors stemming from gradients of H^1 functions but not those related to any holes in the domain. We define the split near-kernel (SNK) coarse space $V_G \subset \mathbf{R}^{\mathbf{n}}$ as the vector space spanned by the sequence $(R_i^T D_i G_i)_{1\le i\le N}$, where $G_i := R_i G$, so that $G \subset V_G$. Here we have split G into contributions on subdomains and note that $\dim V_G > \dim G$ in general; we will see that such a splitting is necessary in numerical results. The coarse space matrix $Z \in \mathbf{R}^{\mathbf{n}\times\mathbf{n_0}}$ is a rectangular matrix whose columns are a basis of V_G.

Note that the space V_G is the equivalent of the Nicolaides coarse space [6] for the Laplace problem or the rigid body modes coarse space for the linear elasticity problem; see [1]. For a non simply connected domain, this coarse space is not large enough. This can be seen by the fact that the kernel of the curl operator is strictly larger than the space of gradients of functions. In order to adequately supplement it and inspired by previous works on the GenEO spectral coarse space [8], [1], we proceed as follows.

Let A_i^{Neu} denote the discretization matrix of a Neumann problem in subdomain i. Let ξ_{0i} denote the b_i-orthogonal projection from $\mathbf{R}^{\#\mathcal{N}_i}$ on G_i parallel to $G_i^{\perp_{B_i}}$ where

$$b_i(\mathbf{U}_i,\mathbf{V}_i) = (R_i A R_i^T \mathbf{U}_i, \mathbf{V}_i)$$

Solve a generalised eigenvalue problem (GEVP) in each subdomain Ω_i

$$\boxed{\begin{array}{c}\text{Find } (\mathbf{V}_{ik},\lambda_{ik}) \in \mathbf{R}^{\#\mathcal{N}_i}\setminus\{0\}\times\mathbf{R} \text{ such that}\\ (I-\xi_{0i}^T)D_iR_iAR_i^TD_i(I-\xi_{0i})\mathbf{V}_{ik} = \lambda_{ik}A_i^{Neu}\mathbf{V}_{ik}\end{array}} \tag{2}$$

Let $V_{i,geneo}^{\tau} \subset \mathbf{R}^{\#\mathcal{N}}$ be the vector space spanned by the family of vectors $(R_i^T D_i(I-\xi_{0i})\mathbf{V}_{ik})_{\lambda_{ik}>\tau}$ corresponding to eigenvalues larger than τ. Let V_{geneo}^{τ} be the vector space spanned by the collection over all subdomains of vector spaces $(V_{j,geneo}^{\tau})_{1\le j\le N}$. Define the coarse space $V_0 := V_G + V_{geneo}^{\tau}$.

Then, we can prove the following theorem:

Theorem 1 (Convergence of additive Schwarz with the split near-kernel and GenEO coarse space): For a given user-defined constant $\tau > 0$, namely the eigenvalue threshold within the above GEVP, the spectral condition number estimate for the two-level additive Schwarz preconditioner

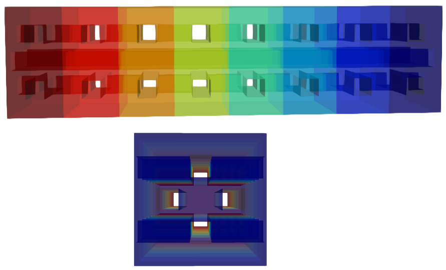

Figure 1. The test configuration with holes for $N = 8$ subdomains, with views from the side (left) and front (right) of the beam. The subdomain partitioning is also shown on the left with subdomains indicated by the different colours.

denoted $M^{-1}_{\mathrm{AS},2}$, with coarse space stemming from V_0, is bounded by

$$\kappa(M^{-1}_{\mathrm{AS},2}A) \leq (1 + k_1\tau)k_0.$$

where k_1 and k_0 are numbers linked to connectivity graph of the domain decomposition (typically the number of neighbours of a subdomain). Note that this estimate does not depend on the number of subdomains nor on the topology of the computational domain. Numerical experiments confirm our theory.

III. Numerical results

All our numerical tests were performed with the free open-source domain specific language FreeFEM [3] and for each test configuration we compare the iteration counts of the following methods:

- AMS: the auxiliary-space Maxwel [5]l solver, originally introduced by Hiptmair and Xu [4] and available via PETSc [2];
- AS: the one-level additive Schwarz preconditioner;
- AS-SNK-GenEO: the proposed two-level Schwarz preconditioner with the adaptive coarse space V_0.

The domain has various holes that extend across the domain. Four square holes extend along the whole length of the beam; see 1 (right). Further, two square holes perpendicularly cross each subdomain, from one lateral side to the opposite side, and also connect to two of the four holes along the length; see 1 (left). Since we perform a weak scalability test, the number of holes increases with the number of subdomains N. Results are gathered in I. As we can see, in this non-convex configuration the AMS preconditioner is no longer robust, resulting in a significant degradation in performance, with iteration counts increasing from 43 to 1302.

In the original work of Hiptmair and Xu [4], the theory requires that the domain is convex which implies, for example, that the kernel of the curl–curl operator is spanned by the gradients of H^1 functions. For non-convex domains such as those with holes as considered in our third test configuration, this no longer holds and the kernel can be larger. Although this larger kernel can be computed [7], this remains computationally expensive and it is unclear how it can be incorporated into the AMS preconditioner.

Table I

A weak scalability study detailing iteration counts required for the (non-convex) homogeneous beam problem with holes and mixed boundary conditions.

N	8	16	32	64	128	256
#dofs	113K	226K	451K	901K	1800K	3600K
NK size	18K	36K	72K	144K	288K	576K
SNK size	24K	49K	99K	198K	397K	794K
GenEO size	18	42	90	186	378	762
AMS	43	67	113	321	588	1302
AS	37	61	106	173	294	557
AS-SNK-GenEO	23	24	25	26	27	27

The one-level AS preconditioner shows a similar lack of robustness in I, with an iteration count reaching as high as 557. In contrast, the full AS-SNK-GenEO preconditioner shows nearly constant iteration counts, increasing from 23 only to 27, and so good robustness. This time, the near-kernel coarse space needs to be enriched and GenEO selects around three vectors per subdomain to enter the coarse space.

References

[1] Victorita Dolean, Pierre Jolivet, and Frédéric Nataf. *An Introduction to Domain Decomposition Methods: algorithms, theory and parallel implementation*. SIAM, 2015.

[2] Satish Balay et al. PETSc/TAO Users Manual. Technical Report ANL-21/39 - Revision 3.20, Argonne National Laboratory, 2023.

[3] F. Hecht. New development in Freefem++. *Journal of Numerical Mathematics*, 20(3-4):251–265, 2012.

[4] Ralf Hiptmair and Jinchao Xu. Nodal auxiliary space preconditioning in $\mathbf{H}(\mathbf{curl})$ and $\mathbf{H}(\mathrm{div})$ spaces. *SIAM Journal on Numerical Analysis*, 45(6):2483–2509, 2007.

[5] Tzanio V Kolev and Panayot S Vassilevski. Parallel auxiliary space AMG for $H(\mathbf{curl})$ problems. *Journal of Computational Mathematics*, pages 604–623, 2009.

[6] Roy A Nicolaides. Deflation of conjugate gradients with applications to boundary value problems. *SIAM Journal on Numerical Analysis*, 24(2):355–365, 1987.

[7] M. Pellikka, S. Suuriniemi, L. Kettunen, and C. Geuzaine. Homology and cohomology computation in finite element modeling. *SIAM Journal on Scientific Computing*, 35(5):B1195–B1214, 2013.

[8] Nicole Spillane, Victorita Dolean, Patrice Hauret, Frédéric Nataf, Clemens Pechstein, and Robert Scheichl. Abstract robust coarse spaces for systems of PDEs via generalized eigenproblems in the overlaps. *Numer. Math.*, 126(4):741–770, 2014.