A robust and adaptive GenEO-type domain decomposition preconditioner for $\mathbf{H}(\text{curl})$ problems in non simply connected three-dimensional geometries

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Abstract—We develop and analyse domain decomposition methods for linear systems of equations arising from conforming finite element discretisations of positive Maxwell-type equations, namely for $\mathbf{H}(\text{curl})$ problems. Convergence of domain decomposition methods rely heavily on the efficiency of the coarse space used in the second level. We design adaptive coarse spaces that complement a near-kernel space made from the gradient of scalar functions. The new class of preconditioner is based on spectral coarse spaces, and is specially designed for curl-conforming discretisations of Maxwell’s equations in non simply connected geometries (i.e. not all loops are reducible to a point).

I. INTRODUCTION

In this work we focus on the efficient solution of the following Maxwell problem:

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{E}) + \gamma \varepsilon \mathbf{E} = \mathbf{f} \quad \text{in } \Omega,$$

$$\mathbf{E} \times \mathbf{n} = 0 \quad \text{on } \partial \Omega.$$  \hspace{1cm} (1)

Here $\mathbf{E}$ is the vector-valued electric field, $\mathbf{f}$ is a source term, while $\mu$ and $\varepsilon$ are electromagnetic parameters which are uniformly bounded and strictly positive but which we allow to be heterogeneous. Further, $\gamma > 0$ is a positive parameter which is allowed to be very small. We suppose $\Omega$ is a computational domain (possibly not simply connected) and $\mathbf{n}$ is the outward normal to $\partial \Omega$. This problem, although positive definite, remains challenging from the solution methods point of view. The reference method to solve this problem, remains up to now, the celebrated algorithm of Hiptmair–Xu from [4], which was identified by the U.S. Department of Energy in 2008 as one of the top ten recent breakthroughs in computational science [5]. For a trivial topology, this algorithm is robust w.r.t. mesh size. But theory and practice (see Table I) show this is not the case for non simply connected domains. This is in contrast with the domain decomposition method we use here which is provably robust for non trivial topologies.

II. COARSE SPACE OF THE TWO-LEVEL SCHWARZ METHOD

Let $A^i_{\text{nec}}$ denote the discretization matrix of a Neumann problem in subdomain $i$. Let $\xi_{0i}$ denote the $b_i$-orthogonal projection from $\mathbb{R}^{n_{1i}}$ on $G_i$, parallel to $G_{1i}^+$. Let

$$b_i(\mathbf{U}_i, \mathbf{V}_i) = (R_i A_i^T U_i, V_i)$$

Solve a generalised eigenvalue problem (GEVP) in each subdomain $\Omega_i$

$$\text{Find } (\mathbf{V}_{ik}, \lambda_{ik}) \in \mathbb{R}^{n_{1i}} \setminus \{0\} \times \mathbb{R} \text{ such that } (I - \xi_{0i}^T) D_i R_i A_i^T D_i (I - \xi_{0i}) \mathbf{V}_{ik} = \lambda_{ik} A_{\text{nec}}^i \mathbf{V}_{ik}.$$  \hspace{1cm} (2)

Let $V_{i,\text{geneo}} \subset \mathbb{R}^{n_{1i}}$ be the vector space spanned by the family of vectors $(R_i^T D_i (I - \xi_{0i}) \mathbf{V}_k)_{\lambda_{ik} > \tau}$ corresponding to eigenvalues larger than $\tau$. Let $V_{\text{geneo}}$ be the vector space spanned by the collection over all subdomains of vector spaces $(V_{j,\text{geneo}})_{1 \leq j \leq N}$. Define the coarse space $V_G := V_G + V_{\text{geneo}}$.

Then, we can prove the following theorem:

Theorem 1 (Convergence of additive Schwarz with the split near-kernel and GenEO coarse space): For a given user-defined constant $\tau > 0$, namely the eigenvalue threshold within the above GEVP, the spectral condition number estimate for the two-level additive Schwarz preconditioner

\[ \frac{\text{cond}(A)}{\text{cond}(A_{\text{nec}})} \leq C_1 \frac{\text{cond}(A)}{\text{cond}(A_{\text{nec}})} + C_2, \]
denoted $M_{AS,2}^{-1}$, with coarse space stemming from $V_0$, is bounded by

$$\kappa(M_{AS,2}^{-1}A) \leq (1 + k_1 \tau) k_0,$$

where $k_1$ and $k_0$ are numbers linked to connectivity graph of the domain decomposition (typically the number of neighbours of a subdomain). Note that this estimate does not depend on the number of subdomains nor on the topology of the computational domain. Numerical experiments confirm our theory.

### III. Numerical Results

All our numerical tests were performed with the free open-source domain specific language FreeFEM [3] and for each test configuration we compare the iteration counts of the following methods:

- AS: the one-level additive Schwarz preconditioner;
- AS-SNK-GenEO: the proposed two-level Schwarz preconditioner with the adaptive coarse space $V_0$.

The domain has various holes that extend across the domain. Four square holes extend along the whole length of the beam; see (right). Further, two square holes perpendicularly cross each subdomain, from one lateral side to the opposite side, and also connect to two of the four holes along the length; see (left). Since we perform a weak scalability test, the number of holes increases with the number of subdomains $N$. Results are gathered in [1] As we can see, in this non-convex configuration the AMS preconditioner is no longer robust, resulting in a significant degradation in performance, with iteration counts increasing from 43 to 1302.

In the original work of Hiptmair and Xu [4], the theory requires that the domain is convex which implies, for example, that the kernel of the curl–curl operator is spanned by the gradients of $H^1$ functions. For non-convex domains such as those with holes as considered in our third test configuration, this no longer holds and the kernel can be larger. Although this larger kernel can be computed [7], this remains computationally expensive and it is unclear how it can be incorporated into the AMS preconditioner.

### References


